

ARNOLD DIFFUSION IN A RESTRICTED PLANAR FOUR-BODY PROBLEM

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ABSTRACT. This paper constructs a certain planar four-body problem which exhibits fast energy growth. The system considered is a quasi-periodic perturbation of the Restricted Planar Circular three-body Problem (RPC3BP). Gelfreich-Turaev's and de la Llave's mechanism is employed to obtain the fast energy growth. The diffusion is created by a heteroclinic cycle formed by two Lyapunov periodic orbits surrounding L_1 and L_2 Lagrangian points and their heteroclinic intersections. Our model is the first known example in celestial mechanics of the a priori chaotic case of Arnold diffusion [1].

1. INTRODUCTION

In this paper, we shall construct solutions of the Restricted Planar four-Body problem (RP4BP) which exhibit long time instabilities, i.e. motions change substantially. The model we employ is that of a Sun-Jupiter-Planet(small)-Asteroid system. In this model the mass of the asteroid is assumed to be negligibly small (in fact zero). The mass of the planet, denoted m_P , is strictly positive and $0 < m_P \ll m_J$ where m_J denotes the mass of Jupiter. The Sun-Jupiter-Planet (S-J-P) system forms a planar three-body problem (P3BP) which has quasi-periodic motion, and the object of study is to understand the motions of the massless asteroid in this system. Gelfreich-Turaev proposed the following mechanism for Arnold diffusion [2]. (It was pointed out to me that R. de la Llave proposed the same mechanism in his unpublished paper [3] and mentioned it in his ICM 2006 talk.) For a Hamiltonian system, $H(p, q, \varepsilon t)$, $q \in \mathbb{T}^d$, $p \in \mathbb{R}^d$, consider the frozen system, $H(p, q, \nu)$. Suppose for each ν , and each energy surface of the energy interval $[h_-, h_+]$ of the frozen Hamiltonian $H(p, q, \nu)$, there are hyperbolic periodic orbits γ_1 and γ_2 with stable and unstable manifolds $W^{u,s}(\gamma_1)$, $W^{u,s}(\gamma_2)$ which make transversal heteroclinic intersections. In [2] adiabatic invariant theory and a special contracting mapping theorem are used to prove that, for a sufficiently small ε , there exists $t_1 > 0$ such that the Hamiltonian

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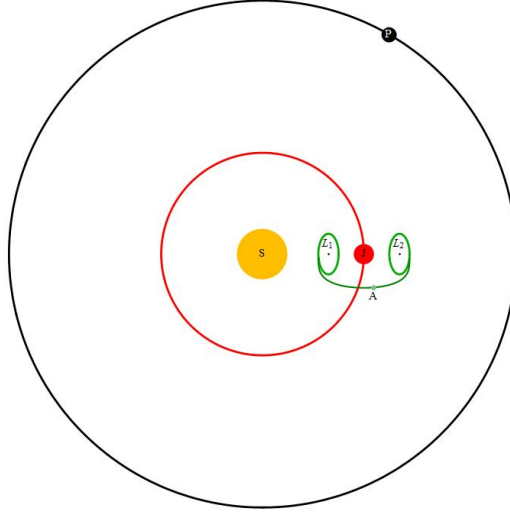


FIGURE 1. Configuration of the four-body problem

$H(p, q, \varepsilon t)$ has a linearly fast diffusing orbit $(p, q)(t)$, i.e.

$$H(p(0), q(0), 0) = h_-, \quad H(p(t), q(t), \varepsilon t) = h_+$$

for some $t \leq t_1/\varepsilon$.

One important feature of the mechanism of [2] is that their methods do not rely on how the Hamiltonian depends on εt . The results hold true for periodic, quasiperiodic and other settings. The key to this paper is applying the [2] mechanism to the RP4BP. In our case, the εt dependence is quasi-periodic.

Notice that, in Planar Restricted Circular three-Body Problem (RPC3BP), there are two normally hyperbolic periodic orbits γ_1, γ_2 surrounding the L_1 and L_2 Lagrangian points respectively and their stable and unstable manifolds have heteroclinic intersections. This gives the “heteroclinic cycle” required in [2].

To obtain slow time-quasi-periodic perturbation we need to exploit the planet. We select S-J-P to have quasi-periodic orbit, so we shall have a quasi-periodic perturbation of the RPC3BP formed by S-J-A. The motions of the asteroid can be described as solutions to Hamilton’s equation with a Hamiltonian H_A that is a quasi-periodic perturbation of the RCP3BP. (see equation (1)):

$$H_A(L_A, \ell_A, G_A, g_A, t) = \text{RPC3BP} + \delta f(L_A, \ell_A, G_A, g_A, \varepsilon t), \quad (L_A, \ell_A, G_A, g_A) \in T^*\mathbb{T}^2.$$

Here we have two independent small parameters. ε is the frequency of the S-J-P system and δ is determined by the mass of the planet m_P . The variables we are

using here are called Delaunay coordinates. See Section 3 and Appendix A for more details. The main theorem proved in this paper is:

Theorem 1. *In the Sun-Jupiter-Planet-Asteroid system, for small ε and $\delta > 0$, there exists a diffusion orbit $(L_A, \ell_A, G_A, g_A)(t)$ with linearly fast speed energy growth. i.e. There is an energy interval $[h_-, h_+]$, such that the energy of the asteroid has growth:*

$$H_A((L_A, \ell_A, G_A, g_A)(0), 0) = h_-, \quad H_A((L_A, \ell_A, G_A, g_A)(t), \varepsilon t) = h_+$$

for some $t \leq t_1/\delta\varepsilon$, for some t_1 , where h_+, h_- are independent of δ and ε .

The problem of the Arnold diffusion is a long story concerning the instability of generic Hamiltonian systems. Here we do not try to mention seas of literatures about Arnold diffusion, but point out the results relevant to our work. Moreover, even though the problem of Arnold diffusion has been studied for half a century. There are scarce concrete examples, esp. in celestial mechanics. As far as the author's knowledge, the only known examples are [14, 15, 16, 17]. Their mechanisms are close to the spirit of Arnold's example, called a priori unstable case.

But our model has new features. The study of energy growth is a simplified version of Arnold diffusion by Mather, so it is also called the Mather problem [1]. The mechanism of diffusion is called "a priori chaotic" by [1], since the reference system has some conserved quantities, but there are orbits which are hyperbolic and with transverse heteroclinic intersections in the manifolds corresponding to the conserved quantities. The systems are not close to integrable, the Nekhoroshev upper bounds for the time of diffusion does not apply. Our model is the first known model of the a priori chaotic case in celestial mechanics. It turns out we obtain linear energy growth rate in our system.

The paper is organized as follows. In Section 2, we give a brief introduction to the Gelfreich-Turaev's mechanism.

In Section 3, we give the construction of the configuration of the four-body problem. We first find a family of quasi-periodic orbits for full S-J-P three-body problem. Here we use Poincaré Continuation method. Then we fix the quasi-periodic orbit and write down the Hamiltonian governing the motion of the asteroid as a quasi-periodic perturbation of the RPC3BP (S-J-A).

In Section 4, the RPC3BP(S-J-A) is studied. There are two normally hyperbolic periodic orbits $\gamma_1(h), \gamma_2(h)$ around the L_1 and L_2 Lagrangian points respectively, on each energy level h for an energy interval $h \in [h_-, h_+]$ ([4]). These periodic orbits $\gamma_1(h)$ and $\gamma_2(h)$ are the key ingredients for the application of Gelfreich-Turaev's mechanism. It is also shown by computer assisted proof in [18] that the stable and unstable manifolds of the normally hyperbolic periodic orbits have transversal heteroclinic intersection.

In Section 5, the heteroclinic cycle of the RPC3BP (S-J-A) is transplanted to the RP4BP. This is done using the hyperbolic theory.

In the last section, Gelfreich and Turaev's mechanism is applied to the RP4BP. With the heteroclinic cycle established in Section 3,4,5, the uniformity assumptions [UA1] [UA2] and the nondegeneracy condition required in [2] are verified. So the main theorem follows.

2. A BRIEF INTRODUCTION TO GELFREICH-TURAEV'S MECHANISM

Consider a Hamiltonian system $H = H(p, q, \varepsilon t)$, $(p, q) \in \mathbb{R}^{2n}$ with ε small. It is routine to consider the frozen system in adiabatic invariant theory $H = H(p, q, \nu)$, where $\nu = \varepsilon t$ is treated as a constant parameter. It is required that the frozen system has a chaotic behavior, namely there exists uniformly-hyperbolic, compact, transitive, invariant set $\Lambda_{h\nu}$ in every energy interval $H \in [h_-, h_+]$ for all ν . In every given energy level, the set $\Lambda_{h\nu}$ is in the closure of a set of hyperbolic periodic orbits each of which has an orbit of a transverse heteroclinic connection to any of the others. This means that orbits of the frozen system may stay close to any of the periodic orbits for an arbitrary number of periods, then come close to another periodic orbit and stay there, and so on. Now we take two periodic families γ_1 and γ_2 of the frozen system. It is shown that under some natural conditions one can arrange jumps between γ_1 and γ_2 in such a way that the energy keeps growing. It is proved that

Theorem 2 (Theorem 2 and 3 of [2]). *Let*

$$v_i(h, \nu) = \frac{1}{T_i} \int_0^{T_i} \frac{H(p, q, \nu)}{d\nu} \Big|_{(p,q)=\gamma_i(t;h,\nu)} dt, \quad i = 1, 2,$$

where T_i is the period of the periodic orbit γ_i . Assume that the differential equation

$$\frac{dh}{d\nu} = \max\{v_1(h, \nu), v_2(h, \nu)\} - \delta' \beta(h, \nu)$$

has a solution $h_\delta(\nu)$ for δ' sufficiently small to suppress β (where β is defined in equation (46) of [2]). Assume the uniformity assumptions [UA1] and [UA2] hold true. Then for all sufficiently small ε the Hamiltonian system has a solution $(p(t), q(t))$ such that

$$H(p(0), q(0), 0) = h_\delta(0), \quad H(p(t), q(t), \varepsilon t) = h_\delta(\varepsilon t).$$

We shall show h_δ grows linearly. Note that the uniformity assumptions [UA1], [UA2] are automatically fulfilled for any compact set of h and ν , which is exactly what we consider. So we do not cite the lengthy formulation of [UA1], [UA2].

3. THE CONFIGURATION OF THE FOUR-BODY PROBLEM

In this section, we first establish the quasi-periodic motion of the Sun-Jupiter-Planet system, then write down the Hamiltonian governing the motion of the asteroid.

3.1. Selection of quasi-periodic orbits of the P3BP formed by Sun-Jupiter-Planet. We need to have a family of quasi-periodic orbits for P3BP (S-J-P) with the following properties:

- (1) Orbits of Sun and Jupiter are nearly circular.
- (2) Orbits of the planet are nearly elliptic, i.e. osculating eccentricity is nearly constant.

The reason that we need a “family” of quasi-periodic orbits is only technical. We need this family to check the nondegeneracy condition in the last section.

We first show the existence of periodic orbits in the S-J-P system when $m_P = 0, m_J \neq 0$, and Sun-Jupiter has circular orbit. i.e. the RPC3BP. Then we continue it to the case of $m_P > 0$.

Before the proof, let us collect the notations in the following definition. Different coordinates will get involved for the convenience of proofs, such as the Cartesian coordinates and Delaunay coordinates. . . . Please go to Appendix A for the derivations and physical meanings of them.

Definition 1. (1) *In the Cartesian coordinates, we use (x, \dot{x}, y, \dot{y}) (or (q, p)). (x, y) (or q) is the position and (\dot{x}, \dot{y}) (or p) is the velocity.*
 (2) *In Delaunay coordinates, we use the variables (L, ℓ, G, g) . L^2 is the semimajor of the Keplerian ellipse, ℓ the mean anomaly, G the angular momentum and g the argument of the perihelion.*

Definition 2. (1) *We use the subscript to indicate the corresponding quantity of a certain body. For example, G_P , the angular momentum of the planet. r_P , the mutual distance from the planet to the origin.*
 (2) *$r_{PJ}, r_{PS}, r_{AJ}, r_{AS}, r_{AP}$ denote the mutual distances between the planet and the Jupiter, the planet and the Sun, the asteroid and the Jupiter, the asteroid and the Sun, the asteroid and the planet, respectively.*

In the following theorem, we prove the existence of a family of periodic orbits for the RPC3BP.

Theorem 3. *The P3BP(S-J-P) has a family of quasiperiodic orbits. This family can be parametrized by the eccentricity of the planet $e_p \in \left[C, 1 - \frac{1 + \alpha}{a_P} \right]$ where C is*

such that the length of the interval $\left[C, 1 - \frac{1 + \alpha}{a_P}\right]$ is $O(1)$ as $m_P, m_J \rightarrow 0$. When $e_P = 1 - \frac{1 + \alpha}{a_P}$, the closest distance between the planet and Jupiter is $\alpha + o(m_P)$ as $m_P \rightarrow 0$. Each of these quasiperiodic orbits covers a torus \mathbb{T}^2 .

Proof. The proof is split into 3 parts.

- 1) Set $m_P = 0$ to find one periodic orbit in the RPC3BP (S-J-P).
- 2) Find a family of periodic orbits in the RPC3BP (S-J-P).
- 3) Continue these orbits to full 3 body problem ($m_P \neq 0$).

STEP 1, find one periodic orbit in the RPC3BP (S-J-P)

In this step, $m_P = 0$. The motion of the planet is a RPC3BP (S-J-P). The Hamiltonian for it in Delaunay coordinates and rotating coordinates is given by (c.f. [7], [8]):

$$H_P = -\frac{1}{2L_P^2} - G_P + \Delta H, \text{ where } \Delta H = \frac{1}{r_P} - \frac{m_J}{r_{PJ}} - \frac{1 - m_J}{r_{PS}}.$$

Here we write the perturbation using the polar coordinates for simplicity. It should be converted into the Delaunay variables. Rotating coordinates is the noninertia coordinates we choose to fix the Sun and Jupiter on the x -axis.

The plan is to consider the double Kepler problem (2BP (S-P) + 2BP (S-J)) by setting $m_J = 0$, then the Hamiltonian in rotating coordinates becomes

$$H_P = -\frac{1}{2L_P^2} - G_P.$$

Consider the resonance relation $g_P = N\ell_P$, where $N = p/q$, $p, q \in \mathbb{Z}$ is a very large rational number. This implies, after p period of the planet, Jupiter experiences q periods. The whole S-J-P system comes back to the original configuration. In the rotating coordinates, this is a periodic orbit for the double Kepler problem. The number N is large so that the period is long. We want to find a periodic solution of the RPC3BP ($m_J > 0$) in a neighbourhood of this resonance. If N is a large number, then the semimajor of the ellipse of the planet should be very large. It is proven by Arenstorf and Barrar that the periodic orbits that are symmetric along the x -axis are locally isolated and therefore can be continued to the RPC3BP for $m_J > 0$.

Lemma 1 (c.f.[9]). *Suppose when $m_J = 0$, $y_P(0) = 0$ and $\dot{x}_P(0) = 0$, i.e. at time $t = 0$, the planet crosses the x -axis perpendicularly, and for a later time $T/2$, the planet again crosses the x -axis perpendicularly, i.e. $y_P(T/2) = 0$, $\dot{x}_P(T/2) = 0$. Then except at most finitely many eccentricity e_P , there exists a periodic orbit of the RPC3BP(S-J-P) for each e_P and for $m_J > 0$ sufficiently small, whose period is slightly different from T , with error $o(m_J)$.*

We use the Keplerian relation $\frac{a^3}{T^2} = \frac{1}{4\pi^2}$ that relates the semimajor a to the period T . Moreover, we have $a = L^2$ relating the Delaunay variable L to the semimajor. Since we consider fixed N , we fix the period T and hence the semimajor a_P and L_P .

STEP 2, find a family of periodic orbits for the RPC3BP (S-J-P)

To find a family periodic orbits for the RPC3BP (S-J-P), we notice in the Kepler problem, the period of the ellipse depends only on the semimajor. Varying the eccentricity, we obtain a family periodic orbits parametrized by e_P in the double Kepler problem without violating the resonance relation $g_P = N\ell_P$. For each one of them, we can use Lemma 1 to continue it to positive m_J . Now we want the closest distance between the planet and Jupiter to be $\alpha > 0$. Let us see how large should e_P be. The orbit of Jupiter is a circle of radius 1 centered at the Sun and that of the planet is an ellipse focused at the Sun. From geometric consideration, we immediately obtain

$$a_P - a_P e_P \geq 1 + \alpha.$$

This gives us $e_P \leq 1 - \frac{1+\alpha}{a_P}$. Now we apply Lemma 1 to $e_P \in \left[0, 1 - \frac{1+\alpha}{a_P}\right]$ with possibly finitely many exceptional e_P 's. We choose some $C \in \left[0, 1 - \frac{1+\alpha}{a_P}\right)$ such that the interval $\left[C, 1 - \frac{1+\alpha}{a_P}\right]$ contains no exceptional e_P 's. Then we can find a uniform m_J for all the e_P because of the compactness of $\left[C, 1 - \frac{1+\alpha}{a_P}\right]$.

STEP 3, continue the family of periodic orbits of the RPC3BP (S-J-P) to the P3BP (S-J-P)

Now we use Poincaré's continuation method to continue the periodic orbits in the RPC3BP to quasi-periodic orbits in the full 3 body problem. The result was first proved in [10], and the proof is reformulated in a concise form in [8].

Lemma 2. (*Theorem 9.6.1 in [8]*)

Any elementary periodic solution of the planar restricted 3 body problem whose period T is not an integer can be continued into the full 3 body problem with one small mass.

Here, "elementary" means, the periodic orbit is "isolated" on the energy surface. Our periodic orbit found in Step 1 has this property.

In order to apply the theorem in our case, we only need to ensure that the period T is not an integer. In fact, We get a periodic orbit of the RPC3BP (S-J-P) from a Keplerian elliptic orbit. In order to avoid the danger that $T \in \mathbb{Z}$, we allow in the resonance relation $g_P = N\ell_P$ that N is a fraction. After continuing a periodic orbit of the RPC3BP to the full 3BP(S-J-P), the orbit is no longer periodic in general

but is quasi-periodic. The dimensional consideration shows that this quasiperiodic orbit covers a torus \mathbb{T}^2 . \square

3.2. The motion of the asteroid. In this part, we consider the motion of the asteroid under the force from the Sun, Jupiter, and the planet. The following theorem establishes that the Hamiltonian describing the motion of the asteroid in the 4BP(S-J-P-A) is quasi-periodic perturbation of the RCP3BP(S-J-A) with 2.5 degrees of freedom. This is the Hamiltonian for which we prove the existence of diffusing orbits for this Hamiltonian.

Theorem 4. *If the motion of S-J-P is chosen to be quasi-periodic, the Hamiltonian of the motion of the asteroid can be written into the form:*

$$(1) \quad H_A(L_A, \ell_A, G_A, g_A, \varepsilon t) = -\frac{1}{2L_A^2} - G_A + \Delta H + \delta f(\ell_A, L_A, g_A, G_A, \varepsilon t),$$

where $(L_A, \ell_A, G_A, g_A) \in T^*\mathbb{T}^2$, the $\delta = o(m_P)$ as $m_P \rightarrow 0$, f is quasi-periodic w.r.t. t , and $\varepsilon = 1/N$ for some large N . i.e. $H_A = \text{RPC3BP} + \text{quasi-periodic perturbation}$.

Proof. The Hamiltonian of the motion of the asteroid can be written in the complete form:

$$H_A(x_A, \dot{x}_A, \varepsilon t) = \frac{1}{2} \dot{x}_A^2 - \frac{1 - m_J}{r_{AS}} - \frac{m_J}{r_{AJ}} - \frac{m_P}{r_{AP}}, \quad (x_A, \dot{x}_A) \in \mathbb{R}^4.$$

The first term is the kinetic energy and the last 3 terms are the potential energy from the Sun, Jupiter and the planet respectively.

We want it to be close to the Hamiltonian of the RPC3BP (S-J-A). The problem is the orbit of the Jupiter is no longer circular, because the mass of the planet is positive. We first choose the mass center of the Sun, Jupiter and the planet as the origin, then write the Hamiltonian in rotating coordinates. We want that, in the rotating coordinates, Sun and Jupiter lie on a line parallel to the x-axis. As a result this rotating coordinates is not uniform. We use the following transformation.

$$\begin{cases} q_A := \exp(\ell_J K) x_A \\ p_A := \exp(\ell_J K) \dot{x}_A \end{cases},$$

where $K = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and ℓ_J is the mean anomaly of the Jupiter after taking into consideration the perturbation by the planet, see Appendix A, equation (12).

$\exp(\ell_J K)$ is an orthogonal matrix, So it is easy to check the following identity.

$$\begin{bmatrix} e^{\ell_J K} & 0 \\ 0 & e^{\ell_J K} \end{bmatrix} \begin{bmatrix} 0 & Id \\ -Id & 0 \end{bmatrix} \begin{bmatrix} e^{\ell_J K} & 0 \\ 0 & e^{\ell_J K} \end{bmatrix}^T = \begin{bmatrix} 0 & Id \\ -Id & 0 \end{bmatrix}.$$

This shows that the change of coordinates is symplectic. The coordinate system (q_A, p_A) is also Cartesian.

$\dot{\ell}_J \neq \text{constant}$, so the rotating angular velocity of the coordinate frame is not uniform. Instead of $-q_A^T K p_A$, the term

$$-\dot{\ell}_J q_A^T K p_A = - \left(1 + m_P \frac{\partial H_1}{\partial L_J} \right) q_A^T K p_A$$

would appear in the new Hamiltonian as the Coriolis term (c.f. the 6th chapter of [8]), where $m_P H_1$ is defined to be the term $\frac{P_1 \cdot P_2}{m_0} - \frac{m_1 m_2}{|Q_1 - Q_2|}$, $i = J, 2 = P$ in equation (11) in Appendix B. Now we get the Hamiltonian:

$$H_A = \frac{1}{2} p_A^2 - \dot{\ell}_J q_A^T K p_A - \frac{1 - m_J}{r_{AS}} - \frac{m_J}{r_{AJ}} - \frac{m_P}{r_{AP}}$$

Sun and Jupiter are very close to the points $(-m_J, 0)$, $(1 - m_J, 0)$. This deviation was explained and estimated in Lemma 3.1.3. Now plug in the terms:

$$-\frac{1 - m_J}{r_1} - \frac{m_J}{r_2} + \frac{1 - m_J}{r_1} + \frac{m_J}{r_2}.$$

(r_1 and r_2 are defined to be the distance from the asteroid to $(-m_J, 0)$, $(1 - m_J, 0)$ respectively. The difference of r_1 to r_{AS} and r_2 to r_{AJ} is $o(m_P)$.)

So the Hamiltonian becomes:

$$H_A = \left[\frac{1}{2} p_A^2 - q_A^T K p_A - \frac{1 - m_J}{r_1} - \frac{m_J}{r_2} \right] + \left[-m_P \frac{\partial H_1}{\partial L_J} q_A^T K p_A \right. \\ \left. + (1 - m_J) \left(\frac{1}{r_1} - \frac{1}{r_{AS}} \right) + m_J \left(\frac{1}{r_2} - \frac{1}{r_{AJ}} \right) - \frac{m_P}{r_{AP}} \right].$$

In this expression, the first bracket is the RPC3BP in a uniform rotating coordinates. The second bracket is of order $o(m_P)$ since it is zero when m_P is zero and the Hamiltonian system depends on m_P analytically. When we plug in the information of the motion of Sun-Jupiter-Planet $(\ell_J, L_J, g_J, G_J, \ell_P, L_P, g_P, G_P)$ as a quasi-periodic vector-valued function of t . The second bracket (denoted by δf) depends on time t quasi-periodically.

The Hamiltonian H_A is written in terms of the Cartesian coordinates (q_A, p_A) . We can convert the Cartesian coordinates into Delaunay coordinates to obtain (1), where the first 3 terms are the Hamiltonian of the RPC3BP and

$$(2) \quad \delta f(\ell_A, L_A, g_A, G_A, \varepsilon t) := -m_P \frac{\partial H_1}{\partial L_J} q_A^T K p_A + (1 - m_J) \left(\frac{1}{r_1} - \frac{1}{r_{AS}} \right) + \left(\frac{m_J}{r_2} - \frac{m_J}{r_{AJ}} \right) - \frac{m_P}{r_{AP}},$$

Here the $\delta = o(m_P)$ as $m_P \rightarrow 0$. We use $\varepsilon = 1/N$ to denote the frequency of the periodic orbits that appear in the proof of Theorem 3 when $m_P = m_J = 0$. \square

Remark 1. For the perturbation (2), we have

- (1) the term $\frac{\partial H_1}{\partial L_J} q_A^T K p_A$ blows up when H_1 tends to infinity. i.e. Jupiter collides with the planet.
- (2) the term $(1 - m_J) \left(\frac{1}{r_1} - \frac{1}{r_{AS}} \right) + m_J \left(\frac{1}{r_2} - \frac{1}{r_{AJ}} \right)$ blows up when each of the denominator tends to zero. i.e. the asteroid collides with the Sun or Jupiter.
- (3) the term $\frac{m_P}{r_{AP}}$ blows up when r_{AP} tends to zero, i.e. the planet collides with the asteroid.

4. EXISTENCE OF HETEROCLINIC CYCLE FOR THE RPC3BP (S-J-A)

In this section we neglect the influence of the planet in the 4BP(S-J-P-A) as defined in above in Section 3. The goal is prove the existence of two normally hyperbolic periodic orbits and the existence of transverse heteroclinic connections between them for the RCP3BP(S-J-A). In Section 5, it will be shown that these phenomena persist to the 4BP(S-J-P-A) when we let $m_P > 0$.

4.1. The existence of two hyperbolic periodic orbits in the RPC3BP. We describe the periodic motions γ_1 (resp. γ_2) of the asteroid near the Lagrangian point L_1 (resp. L_2). Here, it is more convenient for us to study the motion in (x, y, \dot{x}, \dot{y}) Cartesian coordinates. In this coordinates (rotating coordinates), the equations of motion are (c.f.[7]):

$$\begin{cases} x'' - 2\dot{y} = \Omega_x, \\ y'' + 2\dot{x} = \Omega_y, \end{cases} \quad \text{where } \Omega = \frac{x^2 + y^2}{2} + \frac{1 - m_J}{r_1} + \frac{m_J}{r_2} + \frac{m_J(1 - m_J)}{2},$$

$$\text{and } r_1 = \sqrt{(x + m_J)^2 + y^2}, \quad r_2 = \sqrt{(x + m_J - 1)^2 + y^2}.$$

It is easy to check that these equations have an integral—the Jacobi integral.

$$(3) \quad J(x, y, \dot{x}, \dot{y}) := -(\dot{x}^2 + \dot{y}^2) + 2\Omega = -2h,$$

where h is the energy of the RPC3BP. The RPC3BP has 5 equilibria called Lagrangian points (see Figure 2). We want to pay attention to the collinear equilibria L_1, L_2 lying on the x-axis. The L_1 and L_2 Lagrangian points are the two positive critical points of the potential Ω restricted on the x -axis (c.f. Chapter 2.5 of [7]):

$$(4) \quad \frac{x^2}{2} + \frac{1 - m_J}{|x + m_J|} + \frac{m_J}{|x + m_J - 1|}$$

We immediately have the following lemma

Lemma 3. The distance from Jupiter to the Lagrangian points L_1 and L_2 is $O(m_J^{1/3})$ as $m_J \rightarrow 0$.

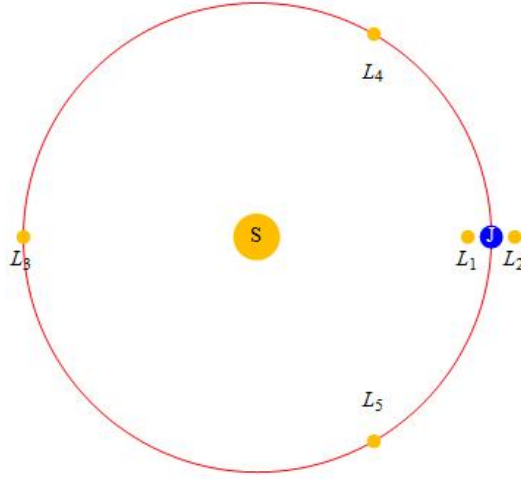


FIGURE 2. Five Lagrangian equilibria of the RPC3BP

Proof. We consider L_2 only, i.e. $x > 1 - m_J$. The L_1 case is similar. Then (4) becomes

$$\frac{x^2}{2} + \frac{1 - m_J}{x + m_J} + \frac{m_J}{x + m_J - 1}.$$

Taking derivatiave and setting the derivative to be 0, we get an equation

$$x - \frac{1 - m_J}{(x + m_J)^2} - \frac{m_J}{(x - 1 + m_J)^2} = 0.$$

As a first step approximation, we suppose $x = 1 - m_J + Cm_J^\alpha$ for some constant C and α and plug it into the equation to get

$$1 - m_J + Cm_J^\alpha(1 - m_J)(1 - 2Cm_J^\alpha) - Cm_J^{1-\alpha} = 0.$$

This equation holds only if $\alpha = 1/3$. □

Since L_1 and L_2 are critical points of the potential (4), we can linearize the Hamiltonian system in a neighbourhood of the two points.

Lemma 4. ([4]) *The linearized systems in a neighbourhood of L_1 and L_2 have eigenvalues $\lambda_{1,2} = \pm\lambda$, $\lambda_{3,4} = \pm i\kappa$, the corresponding eigenvectors, u_1, u_2, w_1, w_2 , and the general solution:*

$$(5) \quad u(t) = \alpha_1 u_1 e^{\lambda t} + \alpha_2 u_2 e^{-\lambda t} + 2 \operatorname{Re}(\beta e^{i\kappa t} w_1).$$

From this Lemma, we can see, for the linearized system, the two complex conjugate purely imaginary eigenvalues give rise to periodic orbit, and the two real eigenvalues give the stable and unstable directions.

It is proved in [4] that:

Theorem 5. ([4]) *The periodic orbits that correspond to $\alpha_1 = \alpha_2 = 0$, project into the $x - y$ -plane as an ellipse with major axis in the direction of y -axis, and minor axis in the direction of the x -axis. The direction on the orbit is counterclockwise. (see Figure 3)*

The proof of the theorem is an application of the Lyapunov center theorem (c.f. [4] and [8], Theorem 9.2.1).

4.2. The existence of heteroclinic intersection in the RPC3BP. To prove the presence of transversal intersections for $W^s(\gamma_1(J))$ and $W^u(\gamma_2(J))$, and vice versa, we cite the result from [18]. In [18], the authors study the Hill problem that is a simplification of the RPC3BP in the limit $m_J \rightarrow 0$. Since m_J is small, we consider the Hill limit $m_J \rightarrow 0$ with proper translation and rescaling. Namely, we consider the following (c.f.[18]).

$$x = Xm_J^{1/3} + m_J - 1, \quad \dot{x} = \dot{X}m_J^{1/3}, \quad y = Ym_J^{1/3}, \quad \dot{y} = \dot{Y}m_J^{1/3}.$$

The rescaling is natural in view of Lemma 3. Then we express J in terms of (X, Y, \dot{X}, \dot{Y}) , to obtain the Jacobi constant in the Hill's limit

$$J_H(X, Y, \dot{X}, \dot{Y}) = -\dot{X}^2 - \dot{Y}^2 + \frac{2}{(X^2 + Y^2)^{1/2}} + 3X^2 + O(m_J^{1/3}),$$

where $J_H = m_J^{-2/3}(J - 3(1 - m_J))$ and $m_J^{2/3}J_H \rightarrow J - 3$ as $m_J \rightarrow 0$.

It is proven in [18] that

Lemma 5 ([18]). *For the Hill problem, for $\frac{1}{2}|J_H|^{-3/2} > 1/18$ the invariant manifolds of the Lyapunov periodic orbits give rise to homoclinic and heteroclinic orbits, connecting a vicinity of L_1 to itself and to one of L_2 and vice versa. These results extend to the RPC3BP for m_J sufficiently small.*

This lemma gives us the heteroclinic intersections between the invariant manifolds of the Lyapunov orbits for RPC3BP(S-J-A) for sufficiently small m_J .

5. EXISTENCE OF HETEROCLINIC CYCLE FOR FOUR-BODY PROBLEM

In the previous sections, we have shown the existence of the heteroclinic cycle for the RPC3BP. Now we want to establish the structure for our four-body problem.

According to the Hamiltonian H_A (1), our four-body problem is a perturbation to the RPC3BP (S-J-A). The RPC3BP (S-J-A) and P4BP (S-J-P-A) have different degrees of freedom and phase spaces. So first we use the classical hyperbolicity theory to show the persistence of the two normally hyperbolic invariant cylinders and persistence of heteroclinic intersections under small perturbation. Second, we show that after freezing the P4BP the perturbation does not destroy the periodic orbits within the two cylinders.

5.1. Translation of Section 4 into the language of classical hyperbolicity theory. Recall that the RPC3BP has 2 degrees of freedom and its phase space is $T^*\mathbb{T}^2$. Denote the flow of the RCP3BP by Φ_t . Fixing the energy restricts the dynamics to a 3 dimensional energy surface denoted by $M(h), h \in [h_-, h_+]$. (In the following sections, we use h instead of the Jacobi constant J to stand for the energy to be consistent with the main theorem, $J = -2h$.) On every energy surface $M(h)$, there are 2 hyperbolic periodic orbits $\gamma_1(h), \gamma_2(h)$ (see Theorem 5).

Definition 3. Define the cylinder $C_i = \bigcup_h \gamma_i(h)$, $i = 1, 2$. These cylinders are two dimensional, and each is diffeomorphism to $[h_-, h_+] \times \mathbb{T}^1$. Additionally their stable and unstable manifolds, denoted $W^s(C_i)$ and $W^u(C_i)$ respectively are both three dimensional. The heteroclinic intersections of these manifolds

$$\Gamma_{12} = \bigcup_h \Gamma_{12}(h) \subset W^s(C_1) \cap W^u(C_2), \quad \Gamma_{21} = \bigcup_h \Gamma_{21}(h) \subset W^u(C_1) \cap W^s(C_2)$$

are two dimensional objects.

The following lemma is a translation of Section 4 into the language of classical hyperbolicity theory.

Lemma 6. ([1]) For some constant $K, \lambda > 0$ (calculated at the end of Section 4.1), and for all $x \in \gamma_i(h)$, we have the decomposition of the tangent space into stable, unstable, and central subspaces.

$$T_x M(h) = E_x^s \oplus E_x^u \oplus T_x \gamma_i(h)$$

with

$$\|D\Phi_t(x)|_{E_x^s}\| \leq K e^{-\lambda t}, \text{ for } t > 0,$$

$$\|D\Phi_t(x)|_{E_x^u}\| \leq K e^{\lambda t}, \text{ for } t < 0,$$

$$\|D\Phi_t(x)|_{T_x \gamma_i(h)}\| \leq K, \text{ for } t \in \mathbb{R}.$$

The stable and unstable manifolds to C_i : $W^{s(u)}(C_i)$, are three dimensional manifolds diffeomorphic to $[h_-, h_+] \times \mathbb{T}^1 \times \mathbb{R}$, and their heteroclinic intersections Γ_{12} and Γ_{21} are both diffeomorphic to $[h_-, h_+] \times \mathbb{R}$.

5.2. Add the perturbation to the RPC3BP (S-J-A). When the perturbation is added, we start to consider the R4BP. The Hamiltonian is (1):

$$H_A = -\frac{1}{2L_A^2} - G_A + \Delta H + \delta f(\ell_A, L_A, g_A, G_A, \varepsilon t) := H_0 + \delta f,$$

where $H_0 := -\frac{1}{2L_A^2} - G_A + \Delta H$ is the RPC3BP(S-J-A). The system as stated is non-autonomous. To apply the mechanism of [2], the frozen the system with $\varepsilon t = \nu$, $\nu \in \mathbb{T}^2$ is considered. Note the frozen system is autonomous and its Hamiltonian can be written in the form:

$$H_A(\ell_A, L_A, g_A, G_A, \nu) = H_0(\ell_A, L_A, g_A, G_A) + \delta f(\ell_A, L_A, g_A, G_A, \nu).$$

Now we are working with an autonomous system with 2 degrees of freedom. For each fixed $\nu \in \mathbb{T}^2$, using the hyperbolicity theory we have the following lemma. It shows that when $\delta \neq 0$ the cylinders along with their stable and unstable manifolds and heteroclinic intersections persist.

Lemma 7. *Assume the Hamiltonian has the form $H_A = H_0 + \delta f(\nu)$, $\nu \in \mathbb{T}^2$ fixed, $H_A \in C^r$, $2 \leq r < \infty$, Then there exists a δ_ν , such that for $|\delta| < \delta_\nu$, the perturbed cylinder $C_{i,\delta}$ is hyperbolic, C^{r-1} diffeomorphic to C_i , locally invariant and is δ -close to the unperturbed cylinder C_i . Its (un)stable manifolds $W^{s(u)}(C_{\delta,i})$ is also δ -close to the unperturbed $W^{s(u)}(C_i)$ in the C^{r-2} sense. ($i=1,2$). Moreover, the compactness of $\mathbb{T}^2(\ni \nu)$ gives us a uniform $\delta^* > 0$ such that the above statement holds for all $\nu \in \mathbb{T}^2$ and $|\delta| < \delta^*$.*

Proof. Direct application of classical hyperbolicity theory, c.f.[1], theorem 4.2 and theorem A.14, A.12. \square

Remark 2. (1) *Since we have the transversal heteroclinic intersections $\Gamma_{12} \subset W^s(C_1) \cap W^u(C_2)$ and $\Gamma_{21} \subset W^u(C_1) \cap W^s(C_2)$, there exists locally unique new heteroclinic intersections: $\Gamma_{\delta,12} \subset W^s(C_{\delta,1}) \cap W^u(C_{\delta,2})$ and $\Gamma_{\delta,21} \subset W^u(C_{\delta,1}) \cap W^s(C_{\delta,2})$. $\Gamma_{\delta,12}$ ($\Gamma_{\delta,21}$) is δ -close to Γ_{12} (Γ_{21}) in the C^{r-2} sense, and that $\Gamma_{\delta,12}$ ($\Gamma_{\delta,21}$) can be parametrized by a C^{r-1} function on Γ_{12} (Γ_{21}) to the extended phase space.*

(2) *The length of the cylinders. i.e. the energy interval $[h_-, h_+]$ may become shorter after the perturbation. Their difference would be of order δ , which is very small compared with $h_+ - h_-$, so we neglect it in the following discussion for simplicity of notations.*

We still need to show the Lyapunov orbits cannot be broken by the perturbation $\delta f(\nu)$. Then we do not have big gaps within the cylinder.

Lemma 8. *Within each perturbed cylinder, the periodic orbits do not break if we add the perturbation δf to RPC3BP.*

Proof. Because our system is frozen, for fixed ν , it is a system of 2 degrees of freedom. As a matter of fact, the persistence of these perturbed periodic orbits can be established from the Lyapunov center theorem ([8], Theorem 9.2.1) in the same way as the existence of Lyapunov orbits. The Lyapunov orbits established in Theorem 5 is a perturbative result from the linearized system Lemma 4. If we treat the $\delta f(\nu)$ and the nonlinear part of the RPC3BP expanded at L_1 or L_2 Lagrangian points as a whole, to perturb the linearized system, we still have a family of Lyapunov periodic orbits due to the same Lyapunov center theorem used by [4] and [8] provided δ is small. Since we have $\nu \in \mathbb{T}^2$, the compactness of \mathbb{T}^2 gives a uniform δ which works for all ν 's. \square

6. APPLY GELFREICH-TURAEV'S MECHANISM TO THE FOUR-BODY PROBLEM

6.1. Verification of the uniformity assumptions in [2]. Notice the P4BP under consideration satisfies the Gelfreich-Turaev's mechanism exactly. Indeed we have shown the existence of two normally hyperbolic periodic orbits and their transversal heteroclinic intersections on every energy surface of the frozen system in the energy interval $[h_-, h_+]$. In order to apply Gelfreich-Turaev's result, we need to check the uniformity assumption [UA1], [UA2] in [2]. We do not want to cite them here. [UA1] is the hyperbolicity requirement, which is given by Conley's result in Section 4.1, while [UA2] is trivially satisfied, since we only consider finite energy interval which has compactness. Now we have proved the Theorem 1 modulo checking the nondegeneracy condition. The time span $t_1/\delta\varepsilon$ can be seen from the following equations (6), (8).

6.2. Verification of nondegeneracy. We break the proof into several steps. First we write the nondegeneracy condition into a form that we are able to check. Then we show the resulting form is analytically dependent on the variables. Then we use the property of analytic functions to show the nondegeneracy holds.

6.2.1. Write integrals responsible for non-degeneracy. In the Hamiltonian (1), we set $\varepsilon t = \nu$. According to Theorem 2, we have

$$(6) \quad \frac{dh}{d\nu} = \max \left\{ \delta \frac{\partial \bar{f}_1(h, \nu)}{\partial \nu}, \delta \frac{\partial \bar{f}_2(h, \nu)}{\partial \nu} \right\} - \delta' \beta(h, \nu)$$

where

$$(7) \quad \bar{f}_i = \frac{1}{T_i} \int_0^{T_i} f|_{\gamma_i}(l_A, L_A, g_A, G_A, \nu) dt,$$

and T_i ($i = 1, 2$) is the period of γ_i . The δ' here is the δ in [2], which is just a small number, much smaller than δ in this paper. So the $\delta'\beta$ term is negligible.

Now, integrate both sides.

(8)

$$h(t_1) - h(0) = \frac{1}{2} \delta(\bar{f}_1(t_1) + \bar{f}_2(t_1) - \bar{f}_1(0) - \bar{f}_2(0)) + \delta \int_0^{t_1} \left| \frac{d}{d\nu}(\bar{f}_1 - \bar{f}_2) \right| d\nu - 2\delta' \int \beta d\nu.$$

So in order to get linear energy growth, our non-degeneracy condition is (Theorem 4 in [2]):

$$\liminf_{t_1 \rightarrow \infty} \frac{1}{t_1} \int_0^{t_1} \left| \frac{d}{d\nu}(\bar{f}_1(\nu) - \bar{f}_2(\nu)) \right| d\nu > 0.$$

In our case, the perturbation is quasi-periodic. So it is sufficient to satisfy:

$$(9) \quad \bar{f}_1(\nu) - \bar{f}_2(\nu) \neq \text{const.}$$

This is also the equation (69) in [2].

6.2.2. *These integrals depend analytically on parameters and variables.* We show the following:

Lemma 9. *The integrals (7) appearing in (9) depend analytically on all the parameters.*

Proof. Originally, away from any collision, the perturbation obtained from Section 3.2 the perturbation δf (equation (2)) is an analytic function in terms of the positions and momenta of Sun, Jupiter and the planet because of the analyticity of the Newton potential.

For the S-J-P system, we can transform the variables into Delaunay coordinates:

$$(\ell_J, L_J, g_J, G_J, \ell_P, L_P, g_P, G_P).$$

The transformation is analytic, so f is analytically dependent on the Delaunay coordinates.

Moreover, once we have chosen a quasi-periodic solution of the S-J-P system, these Delaunay coordinates would be a quasi-periodic function of time t . These quasi-periodic functions are also analytic because they are the solutions of the Hamiltonian equations, which are analytic ODEs (here we use Cauchy's theorem).

So we get the Delaunay coordinates of the S-J-P system are analytically dependent on the time t . \square

6.2.3. *Find terms which should dominate these integrals and become unbounded.*

Lemma 10. *Consider \bar{f}_1 and \bar{f}_2 defined in (7), we have*

$$\bar{f}_1(\nu) - \bar{f}_2(\nu) \neq \text{const},$$

for some e_P satisfying $\left|e_P - \left(1 - \frac{1}{L^2}\right)\right| > c$ where the constant $c > 0$ is independent of m_J, m_P .

Proof. We have shown that $\bar{f}_i(\nu)$ is analytically dependent on ν and e_P . Now consider the Fourier expansion

$$\bar{f}_i(\nu) = \sum_{k \in \mathbb{Z}^2} F_k(e_P) e^{ik\nu}, \quad \nu \in \mathbb{T}^2.$$

To prove the theorem, we only need to show $F_k(e_P), k \in \mathbb{Z}^2 \setminus \{0\}$ does not vanish simultaneously. We know F_k 's dependence on e_P is analytic since $e_P = \sqrt{1 - (G_P/L_P)^2}$. Let us first fix some e_P and vary ν . We want to show the function $\bar{f}_i(\nu)$ has a singularity for some ν , while at some other point ν , it is bounded. This implies, for this fixed e_P , $\bar{f}_i(\nu)$ is not a constant function of ν . So $F_k(e_P), k \in \mathbb{Z}^2 \setminus \{0\}$ does not vanish simultaneously. Each $F_k(e_P)$ is analytic, so the zeros are isolated. We only need to avoid finite possible values of e_P , e.g. the zeros of $F_{(0,1)}(e_P)$ to make $\bar{f}_i(\nu)$ have nonvanishing dependence of ν .

We have found a family of quasi-periodic orbits of S-J-P which can be parametrized by $e_P \in \left[C, 1 - \frac{1+\alpha}{L_P^2}\right]$ in Theorem 3. If we choose $\alpha = \text{dist}(L_2, (1 - m_J, 0)) = O(m_J^{1/3})$ according to Lemma 3, then we expect when $e_P = 1 - \frac{1+\alpha}{L_P^2}$, the planet

passes close to L_2 and is at a distance of $O(m_J^{1/3})$ from Jupiter. We are going to show that for such e_P , the expression $\bar{f}_2(\nu) - \bar{f}_1(\nu)$ blows up. Therefore using the properties of analytic function, we get $\bar{f}_2(\nu) - \bar{f}_1(\nu)$ is not a constant for any choice of e_P . In the end, we choose $e_P = C$. This configuration of P4BP (S-J-P-A) will be away from collisions and give us the desired linear energy growth.

Now let us study the function δf and its averages $\delta \bar{f}_i$. We first consider the term $(1 - m_J) \left(\frac{1}{r_1} - \frac{1}{r_{AS}}\right) + m_J \left(\frac{1}{r_2} - \frac{1}{r_{AJ}}\right)$ in (2). We claim that the contribution of this term to the averages $\delta \bar{f}_i$ is $o(m_P)$ as $m_P \rightarrow 0$. Indeed, we have the estimates

$$(1 - m_J) \left| \left(\frac{1}{r_1} - \frac{1}{r_{AS}} \right) \right| = (1 - m_J) \frac{|r_1 - r_{AS}|}{r_1 r_{AS}} \leq (1 - m_J) \frac{\text{dist}(q_S, (-m_J, 0))}{r_1 r_{AS}} = o(m_P)$$

as $m_P \rightarrow 0$, since the distance between the sun and the point $(-m_J, 0)$ is $\text{dist}(q_S, (-m_J, 0)) = o(m_P)$ as $m_P \rightarrow 0$ and $r_1, r_{AS} = 1 - O(m_J^{1/3})$ as $m_J \rightarrow 0$. We also have

$$m_J \left| \left(\frac{1}{r_2} - \frac{1}{r_{AJ}} \right) \right| = \frac{m_J |r_2 - r_{AJ}|}{r_2 r_{AJ}} \leq \frac{m_J \text{dist}(q_J, (1 - m_J, 0))}{r_2 r_{AJ}} = o(m_P) O(m_J^{2/3})$$

as $m_J, m_P \rightarrow 0$, since the distance between the Jupiter and the point $(1 - m_J, 0)$ is $\text{dist}(q_J, (-m_J, 0)) = o(m_P)$ as $m_P \rightarrow 0$ and $r_2, r_{AJ} = O(m_J^{1/3})$ as $m_J \rightarrow 0$ according to Lemma 3.

Now we consider the term $m_P \frac{\partial H_1}{\partial L_J} q_A^T K p_A$ in (2) where $m_P H_1 = \frac{P_J \cdot P_P}{m_S} - \frac{m_J m_P}{|Q_J - Q_P|}$ in (11). By our choice of e_P , the closest distance between Jupiter and the planet is $O(m_J^{1/3})$. The interaction force between the two bodies is $\frac{m_J m_P (Q_J - Q_P)}{|Q_J - Q_P|^3}$, whose modulus is $m_P O(m_J^{1/3})$. Moreover, the force lasts only for a time interval of order $O(m_J^{1/3})$. Most of the time, the interaction force is $O(m_J m_P)$. This means the motion of the two bodies is only a small perturbation to the double two-body problem. Since p_A, q_A are bounded when we average δf along γ_1, γ_2 . Then the contribution of the term $m_P \frac{\partial H_1}{\partial L_J} q_A^T K p_A$ to the averages $\delta \bar{f}_i$ is $O(m_P m_J^{1/3})$.

Finally, we study the last term in (2), say $\frac{m_P}{r_{AP}}$. We evaluate the average $\frac{1}{T_i} \int_0^{T_i} \frac{1}{r_{AP}} \Big|_{\gamma_i} dt$. We know that γ_1 and γ_2 have almost elliptic orbits from theorem 5. We choose the parametrization $\gamma_i : (L_i + a_i \cos \kappa_i t, b_i \sin \kappa_i t)$, due to expression (5) (Recall the κ_i first appeared in (5)). We obtain

$$\begin{aligned} \frac{1}{T_i} \int_0^{T_i} \frac{1}{r_{AP}} \Big|_{\gamma_i} dt &= \frac{1}{T_i} \int_0^{T_i} \frac{1}{\sqrt{(x_P - L_i - a_i \cos \kappa_i t)^2 + (y_P - b_i \sin \kappa_i t)^2}} dt \\ &= \frac{1}{T_i} \int_0^{T_i} [(x_P - L_i)^2 + y_P^2 + (a_i \cos \kappa_i t)^2 + (b_i \sin \kappa_i t)^2 \\ &\quad - 2a_i(x_P - L_i) \cos \kappa_i t - 2b_i y_P \sin \kappa_i t]^{-1/2} dt. \end{aligned}$$

Denote $(x_P - L_i)^2 + y_P^2$ by $r_{PL_i}^2$, which is the distance between the planet and the L_i Lagrangian point. The integral is taken w.r.t. t , while the position of the planet (x_P, y_P) is independent of t . So (x_P, y_P) has nothing to do with the integral. r_{PL_i} is much larger than the other terms under the square root. So we can use the Taylor expansion.

$$\begin{aligned} &= \frac{1}{r_{PL_i}} - \frac{1}{2r_{PL_i}^3 T_i} \int_0^{T_i} [(a_i \cos \kappa_i t)^2 + (b_i \sin \kappa_i t)^2 - 2a_i(x_P - L_i) \cos \kappa_i t \\ &\quad - 2b_i y_P \sin \kappa_i t] dt + O(1/r_{PL_i}^5) = \frac{1}{r_{PL_i}} - \frac{a_i^2 + b_i^2}{4r_{PL_i}^3} + O\left(\frac{1}{r_{PL_i}^5}\right). \end{aligned}$$

To summarize all the above computations, we obtain that when $e_P \rightarrow 1 - \frac{1 + \alpha}{L_P^2}$ where $\alpha = \text{dist}(L_2, (1 - m_J, 0)) = O(m_J^{1/3})$, then $\delta \bar{f}_2 \sim \frac{m_P}{r_{PL_2}}$ blows up while

$\delta \bar{f}_1 \sim \frac{m_P}{r_{PL_1}}$ remains bounded by $O(m_P m_J^{-1/3})$.

This implies that $\bar{f}_1(\nu) - \bar{f}_1(\nu)$ is nonconstant for this choice of e_P . Due to the properties of analytic functions, $\bar{f}_1(\nu) - \bar{f}_1(\nu)$ is nonconstant for all other choices of e_P . So we set $e_P = C$ such that our 4BP is far from collision. Then the proof of the lemma is completed. \square

APPENDIX A. DELAUNAY VARIABLES

The purpose of the appendix is to give a brief introduction to the Delaunay coordinates used in the paper. The materials could be found in [8]. For two-body problem of the form

$$H(P, Q) = \frac{|P|^2}{2m} - \frac{k}{|Q|}, \quad (P, Q) \in \mathbb{R}^4,$$

we know it is integrable in the Liouville-Arnold sense when $H < 0$. So we have the action-angle variables (L, ℓ, G, g) to write the Hamiltonian can be written as

$$H(L, \ell, G, g) = -\frac{mk^2}{2L^2}, \quad (L, \ell, G, g) \in T^*\mathbb{T}^2.$$

The Hamiltonian equations are

$$\dot{L} = \dot{G} = \dot{g} = 0, \quad \dot{\ell} = \frac{mk^2}{L^3}.$$

If we define the quantities:

E : energy, M : angular momentum, e : eccentricity, a : semimajor, b : semiminor, then we have the following relations which endow the Delaunay coordinates the physical and geometrical meanings.

$$a = \frac{L^2}{mk}, \quad b = \frac{LG}{mk}, \quad E = -\frac{k}{2a}, \quad M = G, \quad e = \sqrt{1 - \left(\frac{G}{L}\right)^2}.$$

Moreover, g is the argument of periapsis and ℓ is called the mean anomaly, and ℓ can be related to the polar angle ψ through

$$\tan \frac{\psi}{2} = \sqrt{\frac{1+e}{1-e}} \cdot \tan \frac{u}{2}, \quad u - e \sin u = \ell.$$

We also have the Kepler's law $\frac{a^3}{T^2} = \frac{1}{(2\pi)^2}$ which relates the semimajor a and the period T of the ellipse. For two-body problem. Consider a body with position (q_1, q_2) and momentum (p_1, p_2) . we have the following formulas

$$\begin{cases} q_1 = a(\cos u - e), \\ q_2 = a\sqrt{1-e^2} \sin u, \end{cases} \quad \begin{cases} p_1 = -\sqrt{mka}^{-1/2} \frac{\sin u}{1 - e \cos u}, \\ p_2 = \sqrt{mka}^{-1/2} \frac{\sqrt{1-e^2} \cos u}{1 - e \cos u}, \end{cases}$$

where u and l are related by $u - e \sin u = \ell$.

Convert everything except u into Delaunay, we have the following

$$(10) \quad \left\{ \begin{array}{l} q_1 = (L^2/mk) \left(\cos u - \sqrt{1 - \frac{G^2}{L^2}} \right), \\ q_2 = (LG/mk) \sin u. \end{array} \right. \quad \left\{ \begin{array}{l} p_1 = -\frac{mk}{L} \frac{\sin u}{1 - \sqrt{1 - \frac{G^2}{L^2}} \cos u}, \\ p_2 = \frac{mk}{L^2} \frac{G \cos u}{1 - \sqrt{1 - \frac{G^2}{L^2}} \cos u}. \end{array} \right.$$

Here g does not enter because the argument of perihelion is chosen to be zero. g will enter if we rotate the (q_1, q_2) and (p_1, p_2) using the matrix $\begin{bmatrix} \cos g & -\sin g \\ \sin g & \cos g \end{bmatrix}$.

APPENDIX B. THREE-BODY PROBLEM IN DELAUNAY VARIABLES

For general three-body problem, the Hamiltonian is

$$H_3(q_0, q_1, q_2, p_0, p_1, p_2) = \sum_{i=0}^2 \frac{p_i^2}{2m_i} - \sum_{0 \leq i < j \leq 2} \frac{m_i m_j}{|q_i - q_j|}$$

where $p_i \in \mathbb{R}^2$ is the momentum, $q_i \in \mathbb{R}^2$ is the position, $m_i > 0$ is the mass. The first sum is the kinetic energy and the second sum is the mutual action potential. The system has 6 degrees of freedom. But it is also translation invariant and rotation invariant. We can eliminate 3 degrees of freedom after reducing the two invariance.

B.1. The first step of symplectic transformation. We first eliminate the translation invariance i.e. the momentum conservation.

$$\sum_{i=0}^2 dp_i \wedge dq_i = d(p_0 + \sum_{i=1}^2 p_i) \wedge dq_0 + \sum_{i=1}^2 dp_i \wedge d(q_i - q_0) := dP_0 \wedge dQ_0 + \sum_{i=1}^2 dP_i \wedge dQ_i$$

We know $dP_0 = 0$ due to momentum conservation.

$$\begin{aligned} H_3 &= \frac{(p_1 + p_2)^2}{2m_0} + \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} - \frac{m_0 m_1}{|Q_1|} - \frac{m_1 m_2}{|q_1 - q_2|} - \frac{m_0 m_2}{|Q_2|} \\ &= \left[\frac{1}{2} \left(\frac{1}{m_0} + \frac{1}{m_1} \right) P_1^2 - \frac{m_0 m_1}{|Q_1|} \right] + \left[\frac{1}{2} \left(\frac{1}{m_0} + \frac{1}{m_2} \right) P_2^2 - \frac{m_0 m_2}{|Q_2|} \right] + \frac{P_1 \cdot P_2}{m_0} - \frac{m_1 m_2}{|Q_1 - Q_2|} \end{aligned}$$

The Hamiltonian of three-body problem is now split into 2 two-body problems with a perturbation.

B.2. The second step of symplectic transformation. We convert the system into Delaunay coordiantes. Our symplectic form becomes

$$\sum_{i=1}^2 dP_i \wedge dQ_i = \sum_{i=1}^2 dL_i \wedge d\ell_i + dG_i \wedge dg_i = d(G_1 + G_2) \wedge dg_1 + dG_2 \wedge d(g_2 - g_1) + \sum_{i=1}^2 dL_i \wedge d\ell_i.$$

The Hamiltonian becomes

$$(11) \quad H_3 = -\frac{k_1}{2L_1^2} - \frac{k_2}{2L_2^2} + \frac{P_1 \cdot P_2}{m_0} - \frac{m_1 m_2}{|Q_1 - Q_2|},$$

where $k_i = \frac{(m_0 m_i)^3}{m_0 + m_i}$, $i = 1, 2$ and we also need to convert P_i, Q_i in the perturbation into Delaunay variables. We notice that in the Hamiltonian, only the relative angle $g_2 - g_1$ appears. The rotation invariance implies the total angular momentum $G_1 + G_2$ is constant. Therefore we eliminate the term $d(G_1 + G_2) \wedge dg_1$ in the symplectic form to obtain a system of three degrees of freedom. Now replace $m_0 = 1 - m_J, m_1 = m_J, m_2 = m_P$ for our S-J-P system. So in our case, the corresponding geometric quantities.

$$L_J^2 = m_J^2(1 - m_J)^2 a_J = m_J^2(1 - m_J)^2, \quad \omega_J = 1,$$

$$L_P^2 = \frac{(1 - m_J)^2 m_P^2}{1 - m_J + m_P} a_P := c a_P,$$

where $c \sim m_P^2$,

$$\omega_P = \frac{k_P}{L_P^3} = \frac{1}{2\pi} \sqrt{\frac{1 - m_J + m_P}{a_P^3}} \sim \frac{1}{2\pi} \sqrt{\frac{1}{a_P^3}}.$$

If we define $m_P H_1 = \frac{P_1 \cdot P_2}{m_0} - \frac{m_1 m_2}{|Q_1 - Q_2|}$ in (11), then the Hamiltonian equations are:

$$(12) \quad \left\{ \begin{array}{l} \dot{\ell}_J = \frac{k_J}{L_J^3} + m_P \frac{\partial H_1}{\partial L_J} = 1 + m_P \frac{\partial H_1}{\partial L_J}, \quad \dot{L}_J = -m_P \frac{\partial H_1}{\partial \ell_J}, \\ \dot{g}_J = m_P \frac{\partial H_1}{\partial G_J}, \quad \dot{G}_J = -m_P \frac{\partial H_1}{\partial g_J}, \\ \dot{\ell}_P = \frac{k_P}{L_P^3} + m_P \frac{\partial H_1}{\partial L_P}, \quad \dot{L}_P = -m_P \frac{\partial H_1}{\partial \ell_P}, \\ \dot{g}_P = m_P \frac{\partial H_1}{\partial G_P}, \quad \dot{G}_P = -m_P \frac{\partial H_1}{\partial g_P}. \end{array} \right.$$

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